MINKOWSKI SUMS OF NEWTON POLYHEDRA IN THE ANALYSIS OF NONLINEAR ALGEBRAIC SYSTEMS

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Abstract. This paper investigates the application of Minkowski sums of Newton polyhedra to the analysis of nonlinear algebraic systems. The geometric properties of Newton polyhedra and methods for constructing reduced systems based on their Minkowski sums are studied. In addition, the significance of this approach for local analysis, asymptotic expansions, and the investigation of the behavior of solutions in the neighborhood of singular points is demonstrated. The obtained results make it possible to simplify the structure of algebraic systems, identify dominant terms, and determine the local properties of solutions. Several examples illustrating the effectiveness of the proposed approach are presented.

Keywords: *Newton polyhedron, Minkowski sum, nonlinear algebraic systems, reduced systems, local analysis, asymptotic expansions, singular points, power geometry.*

1. Introduction

The study of nonlinear algebraic systems is one of the important areas of modern mathematics and has numerous applications in algebraic geometry, mathematical analysis, mechanics, and mathematical physics. In many cases, the direct analysis of such systems is complicated by the presence of nonlinear terms and singular points. Therefore, effective methods for identifying dominant terms and simplifying the structure of systems are of particular interest.

One of the most powerful geometric approaches to the investigation of algebraic systems is the method of Newton polyhedra [1]. This method makes it possible to associate a geometric object with a polynomial or a system of polynomials and to study the local behavior of solutions through the geometric properties of this object. The ideas of Newton polyhedra have been extensively developed in the works of Bruno, Soleev, Batkhin, and other researchers.

When a nonlinear algebraic system consists of several equations, each equation is associated with its own Newton polyhedron [1]. In this case, the Minkowski sum of these polyhedra provides a unified geometric representation of the system and reveals important information about the interaction of dominant monomials [2]. The geometric structure of the Minkowski sum can be used to determine supporting faces, construct normal cones, and derive reduced systems that preserve the principal asymptotic properties of the original system.

The use of Minkowski sums is closely connected with local analysis and power geometry. In particular, the method allows one to investigate asymptotic expansions of solutions near singular points, determine leading-order terms, and construct simplified models describing the local behavior of solutions. Such reduced systems often serve as the first approximation in the study of nonlinear phenomena.

The aim of this paper is to investigate the role of Minkowski sums of Newton polyhedra in the analysis of nonlinear algebraic systems. Special attention is devoted to the construction of reduced systems, the geometric interpretation of normal cones, and the application of these concepts to the local study of singular points and asymptotic expansions.

In recent years, works by Soleev and Batkhin have applied Newton polyhedra and graded geometry methods to the study of algebraic equations and proposed algorithmic approaches for local parametrization problems. In particular, Soleev studied the problem of determining reduced systems near singular points of nonlinear algebraic systems and finding their solutions. In this approach, Newton polyhedra and graded geometry methods are used to analyze algebraic systems.

Soleev considers the following system of algebraic equations

$$f_i(x_1, \dots, x_n) = 0,\ i = 1, \dots, n-1 \qquad (1)$$

To determine the reduced systems of (1) and their corresponding normal cones, the Motzkin-Burger algorithm is applied. This algorithm is based on convex cone theory and allows one to determine the geometric structure of the system and extract its main components. A significant amount of research has been conducted on Newton polyhedra and degree transformations, and these approaches have been further developed in the works of Bruno, Soleev, and Batkhin [3-7].

This paper is a logical continuation of [8-10], where the Newton-Minkowski polyhedron method is used to extract normal cones and corresponding reduced systems for system (1), as well as to study their parametrization. All computations are carried out using the Maple 2021 system [11].

2. The computation process consists of the following steps [8–10]:

Step 1. Determine the supports $D(f_i)$ for each polynomial $f_i(x)$;

Step 2. Construct the Newton polyhedra $\Gamma(f_i)$ for each polynomial $f_i(x)$;

Step 3. Determine the generators of the normal cones $K_j^{(d)}$ corresponding to each Newton polyhedron $\Gamma(f_i)$;

Step 4. For each polynomial $f_i(x)$, find the truncations $\hat{f}_i(x)$ associated with the normal cone $K_j^{(d)}$;

Step 5. Determine the supports $\widehat{D}(f_i)$ for each truncation $\hat{f}_i(x)$;

Step 6. Construct the Newton polyhedra $\hat{\Gamma}(f_i)$ for each truncation $\hat{f}_i(x)$;

Step 7. Compute the Minkowski sum of the Newton polyhedra $\hat{\Gamma}(f_i)$;

Step 8. Construct the common Newton-Minkowski polyhedron of the system;

Step 9. Determine all faces $\hat{\Gamma}_j^{(d)}$ of the Newton-Minkowski polyhedron $\hat{\Gamma}(f)$;

Step 10. Select the points corresponding to each face $\hat{\Gamma}_j^{(d)}$;

Step 11. Formulate the system of linear inequalities for the normal vector P;

Step 12. Solve the resulting system of linear inequalities using the Motzkin–Burger algorithm and determine the generators of the normal cones $\widehat{K}_j^{(d)}$;

Step 13. Extract and analyze the truncated systems corresponding to the determined normal cones $\widehat{K}_j^{(d)}$;

Step 14. Perform the power transformations;

Step 15. Determine the solutions of the system.

In this case, the problem cone is $K = \{p < 0\}$. The local parametric expansions of the algebraic curve near the singular point $x^{(0)}$ are characterized and comprehensively investigated using systems of nonlinear algebraic equations consisting of two equations in three variables and three equations in four variables, respectively:

Example 1.

$$\begin{cases} f_1(x_1, x_2, x_3) \stackrel{\text{def}}{=} a_{11}x_1^4 + a_{12}x_2^4 + a_{13}x_2^2x_3^2 + a_{14}x_2x_3^3 + a_{15}x_3^4 + a_{16}x_1x_2^2x_3^2 = 0 \\ f_2(x_1, x_2, x_3) \stackrel{\text{def}}{=} a_{21}x_1^2 + a_{22}x_2^3 + a_{23}x_3^4 + a_{24}x_1x_2x_3^2 = 0 \end{cases}$$

Example 2.

$$\begin{cases} f_1(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} a_{11}x_1x_2^3 + a_{12}x_1^3x_2 + a_{13}x_2^2x_3 + a_{14}x_3x_4^2 + a_{15}x_1^2x_2^2x_3x_4^2 = 0 \\ f_2(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} a_{21}x_4^2 + a_{22}x_2^3x_3 + a_{23}x_3^4 + a_{24}x_1^5 + a_{25}x_1x_2x_3x_4^5 = 0 \\ f_3(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} a_{31}x_1x_2^2x_3 + a_{32}x_1^3x_3 + a_{33}x_2x_3^4 + a_{34}x_2x_4^4 + a_{35}x_1^2x_2x_3^2x_4 + a_{36}x_1x_2^3x_3^2x_4^2 = 0 \end{cases}$$

3. Results and Discussion

In this study, the Newton polyhedra associated with each polynomial of the nonlinear algebraic system were constructed, and their Minkowski sum was computed. The resulting Newton-Minkowski polyhedron provided a unified geometric representation of the system. The faces of the polyhedron and the corresponding normal cones were identified, allowing the extraction of the associated truncated systems. The generators of the normal cones were determined using the Motzkin-Burger algorithm, and suitable power transformations were applied to simplify the analysis of the system. The obtained results demonstrate that the Newton–Minkowski polyhedron is an effective tool for identifying dominant terms and studying the local behavior of nonlinear algebraic systems in the neighborhood of singular points.

4. Conclusion

A geometric approach based on the Minkowski sums of Newton polyhedra has been presented for the analysis of nonlinear algebraic systems. By constructing the common Newton-Minkowski polyhedron of the system, it becomes possible to identify significant faces, determine the corresponding normal cones, and derive truncated systems that preserve the essential local properties of the original system. The use of the Motzkin-Burger algorithm further facilitates the computation of cone generators and the analysis of local solutions. The results confirm that Minkowski sums provide an effective framework for investigating the structure and local behavior of nonlinear algebraic systems. Future work may focus on extending this approach to higher-dimensional systems and developing efficient computer algebra implementations.

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